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## A METHOD FOR INTEGRATING THE FOKKER-PLANCK EQUATION

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A method for analyzing kinetic processes described by the Fokker-Planck equation is proposed. The method is based on a series expansion of the distribution function in the powers of an evolution operator acting on the equilibrium function (or on the series of temporal derivatives of some parameter). To be specific, the following equation is considered which describes a wide range of effects in standard plasma as well as in a solid-body one:

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial r_i} = \frac{\partial}{\partial v_i} \left( D_{ij} \frac{\partial f}{\partial v_j} - A_i f \right) = \hat{H}f. \quad (1)$$

In the case of a spatially homogeneous distribution function  $f$  of electrons which only depend on the modulus of the velocity vector  $v$  provided the equilibrium temperature  $T_e$  of the electron distribution is known Eq. (1) can be reduced to

$$g \frac{\partial u}{\partial t} = \frac{\partial}{\partial v} \left( gD \frac{\partial u}{\partial v} \right), \quad (2)$$

where  $u = \exp(mv^2/2T_e)f$ ;  $g = v^2 \exp(-mv^2/2T_e)$ ;  $D$  is the diffusion coefficient;  $v$  is the modulus of the velocity vector.

By integrating both sides of (2) twice with respect to  $v$  and assuming the derivative  $\partial u/\partial v$  to be bounded at zero one has

$$u = u_0 + \hat{E}u, \quad (3)$$

where the evolution operation  $\hat{E}$  is defined as follows:

$$\hat{E} = \int_0^v \frac{dv'}{gD} \int_0^{v'} dv'' g \frac{\partial}{\partial t} = \hat{H}^{-1} \frac{\partial}{\partial t}.$$

The operator  $\hat{E}$  now acts on the equilibrium distribution function  $u = u_0$  on the right-hand side of (3); the result is called a quasiequilibrium distribution function (QDF) of the first order. By acting with the operator  $\hat{E}$  on the right-hand side of (3) on the QDF of the first order one finds a QDF of the second order, etc. Thus, the sought distribution function is now represented in the form of an infinite series in the powers of the evolution operator  $\hat{E}$  or in the form of a series of time derivatives of the parameter  $u_0$ :

$$u = \sum_{n=0}^{\infty} \hat{E}^n u_0 = \sum_{n=0}^{\infty} \beta_n \frac{d^{(n)} u_0}{dt^n}, \quad (4)$$

in which the expansion coefficients are found from the formula

$$\beta_n = \int_0^v \frac{dv'}{gD} \int_0^{v'} dv'' g \beta_{n-1}, \quad \beta_0 = 1.$$

Terms with higher derivatives in (4) play an important part only in the initial stages of the procedure. They become smaller in due time, and starting from an instant the expansion (4) can be limited to a finite number of terms. The assumption that an infinite series can be "curtailed" is based on the fact that gradually the system "forgets" the data on the ini-

tial distribution function. In practice the most interesting stage of the procedure is described very often by the QDF of the first order.

The approach described here can be generalized to the continuous spectrum of methods of quasiequilibrium distribution functions developed earlier for analyzing the kinetics of electron levels of groups of atoms [1], or of oscillatory and rotatory molecule levels [2], or of concentrations of multicharge ions, of intensities of Stokes or anti-Stokes components in the forced combination scattering of light [4]. The distribution functions of electrons in discrete or continuous spectra are similar in many respects. The coefficients are given for the series expansion in the derivatives obtained by solving the balance kinetic equations for groupings of electron states of atoms in the single-quantum approximation [1]:

$$\beta_n^i = \sum_{m=i+1}^n \frac{\exp(E_m/T_e)}{g_m \Gamma(m, m-1)} \sum_{k=i}^{m-1} g_k \exp(-E_k/T_e) \beta_k^i.$$

It is obvious that in the case of discrete spectra the integration must be replaced by summation, the diffusion coefficient  $D$  by a constant transition rate from the level  $m$  to  $m-1$ , and also the function  $g$  must be replaced by the Boltzmann factor. Moreover, analogously to the case of discrete spectra [1-4] full use is made of the normalizing condition for the distribution function,

$$\int g u dv = \int_0^{\infty} g u(v, 0) dv. \quad (5)$$

By substituting QDF of the  $m$ -th order on the left an ordinary differential equation in time is obtained of the same order for finding the time dependence for the parameter  $u_0$ . The upper and lower integration limits on the left of (5) can be functions of the parameter  $u_0$  or its derivatives. Therefore, when finding each specific QDF one has to solve its differential equation for the parameter  $u_0$  as a function of time under the conditions  $d^{(n)}u_0/dt^n|_{t \rightarrow \infty} = 0$ ,  $n \neq 0$ .

The choice of  $u_0$  is now illustrated by giving as an example the ordinary heat-conduction equation ( $g = 1$ ,  $D = D_0$ ,  $u(v, 0) = A\delta(v)$ ). By substituting the QDF of the first order on the left of (5), bearing in mind that  $u_0$  is a monotonically decreasing function of time, one obtains after integration over the variation range of  $v$  where  $u > 0$  the following equation for the parameter:

$$\frac{2}{3} u_0 \left( -\frac{2D_0 u_0}{du_0} \right)^{1/2} = A,$$

whose solution differs from the exact expression for  $u_0$  in that  $\sqrt{\pi}$  is replaced by  $4/3$ .

The QDF method is very effective also for analyzing kinetic equations with sources. For example, let us consider the equation

$$\frac{\partial f}{\partial t} = \frac{1}{v^2} \frac{\partial}{\partial v} \left[ v^2 D \left( \frac{\partial f}{\partial v} + \frac{mv}{T} f \right) \right] + q, \quad 4\pi \int_0^{\infty} q v^2 dv = \frac{dN_e}{dt}, \quad (6)$$

where  $q$  is the distribution density of the sources;  $N_e$  is the electron concentration. On the basis of (6) one can describe the action of laser radiation on gas, or the forming of electric charges, or the quasistationary distribution function in a discharge with plasma cathode.

It is assumed that the density of the neutral particle is sufficiently high and that the basic processes which form the distribution function of the electrons with respect to energies are elastic collisions of electrons with heavy particles. The diffusion coefficient is of the form [5]

$$D = d_0 v, \quad d_0 = N_a \sigma T / M,$$

where  $N_a$  is the concentration of the heavy particles;  $T$  is gas temperature,  $M$  is the mass of the heavy particle;  $\sigma$  is the transport cross section of the electron collision with the gas particle.

Our considerations are confined to the case when in the discharge domain electrons of specific energy are generated, that is,  $q = (Q/4\pi v_0^2) \delta(v - v_0)$ ,  $Q = dN_e/dt = \text{const}$ ,  $1/2 m \cdot v_0^2 \gg T$ . By integrating twice over  $v$  on both sides of (6) one finds the QDF of the first order,

$$f \cdot f_0 = u = u_0 + \beta_1 du_0/dt - \gamma Q, \quad f_0 = (\alpha/\pi)^{3/2} \exp(-\alpha v^2), \quad \alpha = m/2T,$$

$$\beta_1 = \frac{1}{4\pi} \int_0^{v_0} \frac{\varphi dv'}{v'^2 D f_0}, \quad \varphi = 4\pi \int_0^{v_0} f_0 v'^2 dv' \leq 1, \quad \gamma = \frac{1}{4\pi} \int_{v_0}^{v_0} \frac{dv'}{v'^2 f_0 D}.$$

For a positive electron source one has  $du_0/dt > 0$ ; therefore,  $u$  in the interval  $0 \leq v \leq v_0$  is an increasing function of  $v$  and a decreasing one in the interval from  $v_0$  to a limiting value  $v_m$  for which the distribution density vanishes. The limiting velocity value  $v_m$  increases monotonically with time from  $v_0$  to infinity. The change in time of  $u_0$  and  $v_m$  is found by solving the following system of two equations:

$$\begin{aligned} u_0 + \beta_1(v_m) du_0/dt - \gamma(v_m) Q &= 0, \\ 4\pi \int_0^{v_m} f_0 \left( u_0 - \beta_1 \frac{du_0}{dt} - \gamma Q \right) v^2 dv &= N_e. \end{aligned} \quad (7)$$

By analyzing the system (7) it can be concluded that under the condition

$$\frac{Q}{\alpha^{1/2} \alpha_0} \ll N_e \ll \frac{1}{8\pi^{3/2} \alpha_0} \frac{\exp(\alpha v_m^2)}{v_m^3}$$

and for time instants starting at which one has  $(v_m/v_0)^4 \ll \exp[\alpha(v_m^2 - v_0^2)]$ , these formulas are valid:

$$u_0 = N_e, \quad du_0/dt = Q.$$

Thus, the quasistationary distribution function can in this case be represented as a sum of two terms, one being the Maxwell distribution with variable electron density  $N_e$ , the other a stationary distortion of the distribution function due to the source. It is noted that a similar assumption with regard to the quasistationary distribution function was previously used in [6, 7] when analyzing the rotatory or oscillatory relaxation of molecules.

The solution procedure in the case of multidimensional Fokker-Planck equations is similar. One then has to find the inverse operator  $\hat{H}^{-1}$ . The solution of the problem if one also takes into account the spatial inhomogeneity can be represented in the form of a series of powers of the operator

$$\hat{E} = \hat{H}^{-1} \left( \frac{\partial}{\partial t} + v_i \frac{\partial}{\partial r_i} \right).$$

Our attention was focused above on the main principles of the method. According to the described scheme one can find a QDF of any order and any limits of its applicability for the electron distribution in plasma. The possibilities are thus opening up now for solving a number of applied problems which are too involved for standard analysis.

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